

An Introduction to Quantum Bayesian Networks for Mixed States

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Abstract

This paper is intended to be a pedagogical introduction to quantum Bayesian networks (QB nets), as I personally use them to represent mixed states (i.e., density matrices, and open quantum systems). A special effort is made to make contact with notions used in textbooks on quantum Shannon Information Theory (quantum SIT), such as the one by Mark Wilde (arXiv:1106.1445)

1 Introduction

This paper is intended to be a pedagogical introduction to quantum Bayesian networks (QB nets), as I personally use them to represent mixed states (i.e., density matrices, and open quantum systems). A special effort is made to make contact with notions used in textbooks on quantum Shannon Information Theory (quantum SIT), such as the one by Mark Wilde[1].

QB nets are a generalization of classical Bayesian networks (CB nets) to quantum mechanics. CB nets have been hot topic in AI circles since the seminal work of Judea Pearl and collaborators that started in the 1980's. A very complete book on CB nets is the one by Koller and Friedman[2].

Just like mankind has devised many names for the idea of God, there are other names for CB nets and variations of the idea. Others have called them causal probabilistic diagrams, factor graphs, probabilistic system diagrams, etc. To be sure, there are some differences between some of these diagrams and CB nets, but all seem to be striving to conjure up the same divine concept. Some of the close siblings of CB nets are discussed in Ref.[3], an IEEE magazine article by Loeliger.

One variation on the CB net idea involves using graphs in which the arrows (a.k.a. directed edges) represent tensor indices and the boxes (a.k.a. nodes, vertices) represent transition matrices. In this approach, call it the tensor-graphs approach, each arrow coming out of a fixed node carries different stuff. In the CB nets approach, the nodes again represent transition matrices, but the arrows perform a very different job. Each arrow coming out of a fixed node carries the same stuff, namely the name of the node the arrow originates at. This difference might appear subtle or even insignificant to the untrained eye, but Bayesian network believers (like me) swear by it, claiming that it is clearer and more powerful than the tensor-graphs approach *when dealing with probabilities*. Bayesian network believers think that using tensor graphs to describe probability networks is like trying to fill round holes with square pegs. Even when the pegs fit, they don't do a very good job.

Classical information theorists have been using tensor-graphs in their field for a long time. See, for example, the chapter on “network information theory” in the book by Cover and Thomas, Ref.[4]. Or look at the book by El Gamal and Kim, Ref.[5], which is devoted exclusively to the subject of network information theory.

Quantum information theorists have been using tensor-graphs in their field for a long time too, at least since the seminal work by Schumacher and collaborators. For an early example of a paper by Schumacher that uses tensor-graphs, see, for example, Ref.[6], published in 1996.

Some early quantum information papers, for example the seminal paper Ref.[7] by Bennett et al. also use a version of tensor-graphs, but they use them in a very loose, ambiguous, imprecise way. Ref.[7] even has some diagrams that sound like sacrilege to the ears of this Bayesian nets believer, such as diagrams that have nodes intended to represent buckets of sewage (Fig.13 in Ref.[7]).

Not only do adherents to “the QB net way” espouse being good parents by treating all arrows (= children) coming out of a fixed node (= parent) the same. We are also very strict about our nodes. Each node has a numerical “value”, and the whole graph also has a value which equals the product of the node values. Nodes aren’t there just for decoration, or as mere labels with no numerical value assigned to them, or only to convey an abstract notion like “do this operation now”.

Many quantum information papers don’t use diagrams at all. They specify their quantum “protocols” or algorithms in terms of “pseudo code”. In my opinion, those papers would be much clearer if they described their algorithms using both, pseudo-code and QB nets, whenever this is possible. The recent book on quantum information theory by Mark Wilde[1] earns high marks in this regard, as it uses a lot of diagrams. Wilde’s diagrams also have a fairly precise meaning. However, they are tensor-graphs, not the wonderful QB nets.

My own work on QB nets started about 15 years ago with Ref.[8]. In that early paper, I dealt only with QB nets for pure quantum states. I’ve been using QB nets for mixed states since at least Ref.[9]. This paper repeats some of the ideas of Ref.[9] for QB nets of mixed states, with (hopefully) some small improvements.

I’ve also written a Mac application that does QB nets called Quantum Fog.¹

I also have a blog called Quantum Bayesian Networks (Ref.[11]) in which I regularly post articles about Bayesian networks and quantum computing.

Subsequent to Refs.[8, 9], other workers have devised their own types of diagrams for doing quantum information theory. Their diagrams are very different from the QB nets in this paper.

- (a) Leifer and Poulin in Ref.[12], and later Leifer and Spekkens in Ref.[13], postulate some directed acyclic graphs, but *they assign a whole density matrix to each node of the graph*. Furthermore, their node density matrices would be quite hard to calculate in practice, especially for complicated graphs. In comparison, a whole QB net is used to describe one density matrix. And the transition matrix that a QB net assigns to each node comes from the definition of a probability amplitude, a really basic thing that requires almost no calculation— certainly less calculation than the node density matrices of Leifer and coworkers.

I like to think of QB nets as: A light container of data, useful as a data structure in computer programming. A vehicle rather than a destination. A transparent, tidy way of organizing a lot of data in a pictorial way, *prior* to intensive calculation, not as being itself the outcome of a major calculation.

¹ I stopped Quantum Fog development in 2006. The application is still available for free at Ref.[10]. Quantum Fog’s last version (Version 2.0) is known to work with Mac OS X ≤ 10.4 . It probably works with some higher versions of Mac OS X too. Quantum Fog only does QB nets for pure states. That’s not because QB nets can’t deal with mixed states as some people think. It’s only because I stopped developing Quantum Fog before I had a chance to add to it the capability to do mixed state calculations.

- (b) Coecke (Ref.[14]) and collaborators use category theory to define their diagrams. In comparison, QB nets are much less abstract. Defining them requires no category theory, just standard, run-of-the-mill quantum mechanics.

The QB nets in this paper are much simpler than the diagrams of (a) and (b). Simplicity can be a virtue in mathematics (consider, for example, abstract algebra’s definition of a group, which is simplicity itself). QB nets are, however, *complicated enough* to be very expressive and useful; that is, they allow one to express numerous quantum mechanical concepts in a useful, practical and enlightening way.

QB nets are a very parsimonious extension of CB nets to quantum mechanics. That is, the definition of QB nets is the smallest possible modification of the definition of CB nets that I can come up with, but enough of a modification so that one can do proper quantum mechanics with them. Keeping QB nets close to CB nets can be very fruitful, because much is already known about CB nets. And CB nets use classical probability so we can sharpen our classical understanding of a problem with them and then try to extrapolate that understanding to QB nets. QB nets retain the same structure as CB nets and can be reduced to them very easily, simply by applying the dephasing operator “cl” (defined below) to each node. Thus, QB nets make the connection to the classical case very direct and explicit.

2 Basic Notation

As usual, $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ will denote the integers, real numbers, and complex numbers, respectively. For $a, b \in \mathbb{Z}$ such that $a \leq b$, let $Z_{a,b} = \{a, a+1, a+2, \dots, b\}$.

Let $\delta_y^x = \delta(x, y)$ denote the Kronecker delta function; it equals 1 if $x = y$ and 0 if $x \neq y$.

For any matrix $M \in \mathbb{C}^{p \times q}$, M^* will denote its complex conjugate, M^T its transpose, and $M^\dagger = M^{*T}$ its Hermitian conjugate.

Random variables² will be denoted by underlined letters; e.g., \underline{a} . The (finite) set of values (states) that \underline{a} can assume will be denoted by $S_{\underline{a}}$. Let $N_{\underline{a}} = |S_{\underline{a}}|$.

The probability that $\underline{a} = a$ will be denoted by $P(\underline{a} = a)$ or $P_{\underline{a}}(a)$, or simply by $P(a)$ if the latter will not lead to confusion in the context it is being used. We will use $pd(S_{\underline{a}})$ to denote the set of all probability distributions with domain $S_{\underline{a}}$.

In quantum physics, \underline{a} has a *fixed, orthonormal* basis $\{|a\rangle_{\underline{a}} : a \in S_{\underline{a}}\}$ associated with it. The vector space spanned by this basis will be denoted by $\mathcal{H}_{\underline{a}}$. Other spans of $\mathcal{H}_{\underline{a}}$ that are not necessarily orthonormal will be denoted by Greek letters with

²We will use the term “random variables” in both classical and quantum physics. Normally, random variables are defined only in classical physics, where they are defined to be functions from an outcome space to a range of values. For technical simplicity, here we define a random variable \underline{a} , in both classical and quantum physics, to be merely the label of a node in a graph, or an n-tuple \underline{x}_K of such labels. Each node or random variable of a CB or QB net is akin to a spacetime event or a collection of them.

subscripts as in $\{|\psi_j\rangle_{\underline{a}}\}_{\forall j}$. In quantum physics, instead of probabilities $P(\underline{a} = a)$, we use “probability amplitudes” (or just “amplitudes” for short) $A(\underline{a} = a)$ (also denoted by $A_{\underline{a}}(a)$ or $A(a)$). In place of $P(a) \geq 0$ and $\sum_a P(a) = 1$, one has $\sum_a |A(a)|^2 = 1$. Besides probability amplitudes, we also use density matrices. A density matrix $\rho_{\underline{a}}$ is a Hermitian, non-negative, unit trace, square matrix (or the associated linear operator) acting on $\mathcal{H}_{\underline{a}}$. We will use $dm(\mathcal{H}_{\underline{a}})$ to denote the set of all density matrices acting on $\mathcal{H}_{\underline{a}}$.

If $\rho_{\underline{x},\underline{a}} \in dm(\mathcal{H}_{\underline{x},\underline{a}})$, and $\rho_{\underline{x}} = \text{tr}_{\underline{a}}(\rho_{\underline{x},\underline{a}}) = \sum_a \langle a | \rho_{\underline{x},\underline{a}} | a \rangle_{\underline{a}} \in dm(\mathcal{H}_{\underline{x}})$, we will say that $\rho_{\underline{x}}$ is a **partial trace of $\rho_{\underline{x},\underline{a}}$** . Given a density matrix $\rho_{\underline{x}_1,\underline{x}_2,\underline{x}_3,\dots} \in dm(\mathcal{H}_{\underline{x}_1,\underline{x}_2,\underline{x}_3,\dots})$, its partial traces will be denoted by omitting its subscripts for the random variables that have been traced over. For example, $\rho_{\underline{x}_2} = \text{tr}_{\underline{x}_1,\underline{x}_3} \rho_{\underline{x}_1,\underline{x}_2,\underline{x}_3}$.

Sometimes, when two random variables $\underline{a}\langle 1 \rangle$ and $\underline{a}\langle 2 \rangle$ satisfy $S_{\underline{a}\langle 1 \rangle} = S_{\underline{a}\langle 2 \rangle}$, we will omit the indices $\langle 1 \rangle$ and $\langle 2 \rangle$ and refer to both random variables as \underline{a} . We shall do this sometimes even if the random variables $\underline{a}\langle 1 \rangle$ and $\underline{a}\langle 2 \rangle$ are not identically distributed! This notation, *if used with caution*, does not lead to confusion and does avoid a lot of index clutter.

When we want to make explicit that an operator Ω maps states in $\mathcal{H}_{\underline{a}}$ to states in $\mathcal{H}_{\underline{b}}$, we will indicate this with a subscript (or superscript) as $\Omega_{\underline{b} \leftarrow \underline{a}}$ or as $\Omega_{\underline{b}|\underline{a}}$. In cases where $S_{\underline{b}} = S_{\underline{a}}$, we will sometimes write $\Omega_{\underline{a}}$ instead of the clearer but longer $\Omega_{\underline{a} \leftarrow \underline{a}}$ or $\Omega_{\underline{a}|\underline{a}}$.

The tensor product symbol \otimes will often be omitted. Sometimes, when two vectors are being tensored, we will list the two vectors vertically instead of horizontally (the latter is more common in the literature). For example, we might write

$$|\phi\rangle_{\underline{a}} \otimes |\psi\rangle_{\underline{b}} = |\phi\rangle_{\underline{a}} |\psi\rangle_{\underline{b}} = \begin{array}{c} |\phi\rangle_{\underline{a}} \\ |\psi\rangle_{\underline{b}} \end{array} . \quad (1)$$

This doesn’t lead to confusion as long as we indicate what vector space each vector lives in. (In the above example, $|\phi\rangle_{\underline{a}}$ clearly lives in $\mathcal{H}_{\underline{a}}$ and $|\psi\rangle_{\underline{b}}$ in $\mathcal{H}_{\underline{b}}$).

In this paper, we consider networks (graphs) with N nodes. Each node is labeled by a random variable \underline{x}_j , where $j \in Z_{1,N}$. For any $J \subset Z_{1,N}$, the ordered set of random variables $\underline{x}_j \ \forall j \in J$ (ordered so that the integer indices j increase from left to right) will be denoted by \underline{x}_J . For example, $\underline{x}_{\{2,4\}} = (\underline{x}_2, \underline{x}_4)$. We will often call the values that \underline{x}_J can assume x_J . For example, $x_{\{2,4\}} = (x_2, x_4)$. We will often abbreviate $\underline{x}_{Z_{1,N}}$ by just \underline{x} . We will often call the values that \underline{x} can assume x .

3 The Sandbox and its Dual

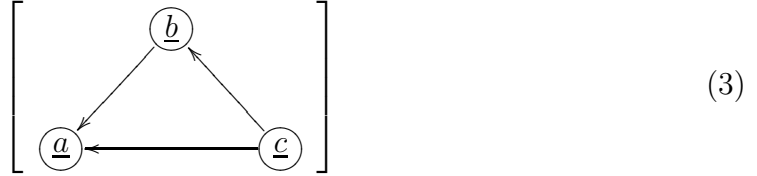
For any expressions $\Omega(x)$ and ρ for which this makes sense, we will use the shorthand notation:

$$\left[\Omega(x) \right] \rho \left[\begin{array}{c} \text{h.c.} \\ x \rightarrow x' \end{array} \right] = \left[\Omega(x) \right] \rho \left[\Omega^\dagger(x') \right] . \quad (2)$$

Here “h.c.” is an abbreviation of “hermitian conjugate”. We will usually use this notation with $\rho = 1$. This notation is especially useful when $\Omega(x)$ is a long expression and we want to avoid writing it twice. We will refer to the space inside the set of square brackets to the left (resp., right) of ρ as **the sandbox** (resp., **its dual or mirror sandbox**).

4 The Meta State

QB nets for pure quantum states were first defined in Ref.[8]. A QB net for a pure state consists of a directed acyclic graph (DAG) and a transition matrix (a complex matrix) assigned to each node of the graph. The transition matrices must satisfy certain requirements. An example of such a pure state QB net is:



If $a \in S_{\underline{a}}, b \in S_{\underline{b}}, c \in S_{\underline{c}}$, $A_{\underline{a}|\underline{b},\underline{c}}(a|b,c)$ is the transition matrix associated with node \underline{a} , $A_{\underline{b}|\underline{c}}(b|c)$ is the transition matrix for node \underline{b} , and $A_{\underline{c}}(c)$ is the transition matrix for node \underline{c} . We must have

$$\sum_a |A_{\underline{a}|\underline{b},\underline{c}}(a|b,c)|^2 = 1, \quad (4a)$$

$$\sum_b |A_{\underline{b}|\underline{c}}(b|c)|^2 = 1, \quad (4b)$$

$$\sum_c |A_{\underline{c}}(c)|^2 = 1. \quad (4c)$$

Define the total probability amplitude $A_{\underline{a},\underline{b},\underline{c}}(a,b,c)$ by

$$A_{\underline{a},\underline{b},\underline{c}}(a,b,c) = A_{\underline{a}|\underline{b},\underline{c}}(a|b,c)A_{\underline{b}|\underline{c}}(b|c)A_{\underline{c}}(c). \quad (5)$$

Note that Eqs.(4) imply

$$\sum_{a,b,c} |A_{\underline{a},\underline{b},\underline{c}}(a,b,c)|^2 = 1. \quad (6)$$

Henceforth, we will sometimes omit the node subscripts from the probability amplitudes. For example, we might use $A(a|b,c)$ instead of $A_{\underline{a}|\underline{b},\underline{c}}(a|b,c)$, if no confusion will arise. This is analogous to probability theory, where we often use $P(a|b,c)$ instead of $P_{\underline{a}|\underline{b},\underline{c}}(a|b,c)$ or $P(\underline{a} = a|\underline{b} = b, \underline{c} = c)$ for a probability.

More generally, suppose the graph has N nodes $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_N$. For $j \in Z_{1,N}$, a node \underline{x}_j with possible states $x_j \in S_{\underline{x}_j}$ and with parent nodes $\underline{x}_{pa(\underline{x}_j)}$ where $pa(\underline{x}_j) \subset Z_{1,N}$, has a transition matrix $A(x_j|x_{pa(\underline{x}_j)})$ which satisfies

$$\sum_{x_j \in S_{\underline{x}_j}} |A(x_j|x_{pa(\underline{x}_j)})|^2 = 1 \quad (7)$$

for all $x_{pa(\underline{x}_j)} \in S_{\underline{x}_{pa(\underline{x}_j)}}$. Let $x. = (x_1, x_2, \dots, x_N)$. If the total amplitude $A(x.)$ is defined by

$$A(x.) = \prod_{j \in Z_{1,N}} A(x_j|x_{pa(\underline{x}_j)}) , \quad (8)$$

then Eqs.(7) imply

$$\sum_{x.} |A(x.)|^2 = 1 . \quad (9)$$

Given any transition matrix of the form $A(r|\vec{c})$, call r the row index and \vec{c} the column indices. Call all $A(r|\vec{c})$ entries with any r but fixed \vec{c} , a column vector of the transition matrix. Eqs.(4) say that each column vector of a transition matrix is normalized. The column vectors may also be mutually orthogonal, in which case we say that the column vectors are orthonormal. For example, the transition matrix for node \underline{a} in the QB net above might also satisfy:

$$\sum_a \left[A_{\underline{a}|\underline{b},\underline{c}}(a|b,c) \right] \begin{bmatrix} \text{h.c.} \\ b,c \rightarrow b',c' \end{bmatrix} = \delta_b^{b'} \delta_c^{c'} . \quad (10)$$

The isometry nodes defined below are another example of a case where the column vectors of the transition matrix are orthonormal. In general, it is not necessary that the column vectors be orthonormal. For example, for the marginalizer nodes defined below, they aren't. What is always necessary is that the total amplitude be normalized, as in Eq.(6), so as to enforce the “unitarity” of quantum mechanics.

The meta state of a QB net was first defined in Ref.[9]. A **meta ket state** is a pure quantum state represented as a ket or as a QB net. For example:

$$|\psi_{\text{meta}}\rangle_{\underline{a},\underline{b},\underline{c}} = \left[\begin{array}{c} \text{Diagram: A triangle with nodes } \underline{a}, \underline{b}, \underline{c} \text{ at the vertices. Arrows point from } \underline{b} \text{ to } \underline{a} \text{ and } \underline{c}, \text{ and from } \underline{c} \text{ to } \underline{a}. \end{array} \right] \quad (11a)$$

$$= \sum_{a,b,c} A_{\underline{a},\underline{b},\underline{c}}(a,b,c) \begin{array}{l} |a\rangle_{\underline{a}} \\ |b\rangle_{\underline{b}} \\ |c\rangle_{\underline{c}} \end{array} , \quad (11b)$$

where $A(a,b,c)$ is defined by Eq.(5). We assume the states $\{|a\rangle_{\underline{a}}\}_{\forall a}$ are orthonormal, and likewise for $\{|b\rangle_{\underline{b}}\}_{\forall b}$ and $\{|c\rangle_{\underline{c}}\}_{\forall c}$. Note that each node of the QB net has its own ket and its own index that is summed over (i.e., bound). For example, node \underline{b} has ket $|b\rangle_{\underline{b}}$ and bound index b .

The projection operator of a meta ket defines a density matrix which we will call the **meta density matrix** of the protocol under consideration. For example, the meta density matrix of the above meta ket is given by:

$$(\rho_{\text{meta}})_{\underline{a}, \underline{b}, \underline{c}} = [|\psi_{\text{meta}}\rangle_{\underline{a}, \underline{b}, \underline{c}}] [\text{h.c.}] \quad (12a)$$

$$= \left[\begin{array}{c} \text{Diagram: A triangle with nodes } \underline{a}, \underline{b}, \underline{c} \text{ in circles. Arrows point from } \underline{b} \text{ to } \underline{a} \text{ and } \underline{c}, \text{ and from } \underline{c} \text{ to } \underline{a}. \\ \end{array} \right] [\text{h.c.}] . \quad (12b)$$

5 Generic Nodes

We find it convenient to define certain special, generic types of nodes.

Marginalizer nodes were first defined in Ref.[8]. In the current version of Quantum Fog, marginalizer nodes are usually denoted by black bullets, whereas non-marginalizer nodes are denoted by larger colored circles. In this paper, we will represent marginalizer nodes by writing a small delta near them. This node “decoration” or subscript is easy to draw by hand and also easy for the eye to spot. For example:

$$|\psi\rangle = \left[\begin{array}{c} \text{Diagram: A central node } (\underline{a}_{\langle 1 \rangle}, \underline{b}_{\langle 1 \rangle}) \text{ in a circle. Two arrows point from it to two nodes above and below. The top node is } \underline{a}_{\langle 2 \rangle} \text{ with a } \delta \text{ next to it. The bottom node is } \underline{b}_{\langle 2 \rangle} \text{ with a } \delta \text{ next to it.} \\ \end{array} \right] , \quad (13)$$

where $\underline{a}_{\langle 1 \rangle}$ and $\underline{a}_{\langle 2 \rangle}$ have the same state space, call it $S_{\underline{a}}$. Likewise, $\underline{b}_{\langle 1 \rangle}$ and $\underline{b}_{\langle 2 \rangle}$ have the same state space, call it $S_{\underline{b}}$. Note that the subscripts $\langle 1 \rangle$ and $\langle 2 \rangle$ are acting like a “time” index along a sort of timeline. For all $a, a' \in S_{\underline{a}}$ and $b, b' \in S_{\underline{b}}$,

$$\begin{aligned} A_{\underline{a}_{\langle 2 \rangle} | \underline{a}_{\langle 1 \rangle}, \underline{b}_{\langle 1 \rangle}}(a | a', b') &= \delta_a^{a'} \\ A_{\underline{b}_{\langle 2 \rangle} | \underline{a}_{\langle 1 \rangle}, \underline{b}_{\langle 1 \rangle}}(b | a', b') &= \delta_b^{b'} \end{aligned} \quad (14)$$

Thus some column vectors of a marginalizer node are equal to each other. To avoid index clutter, we will sometimes omit the indices $\langle 1 \rangle$ and $\langle 2 \rangle$ from the graph of Eq.(13), and draw instead the following graph:

$$|\psi\rangle = \left[\begin{array}{c} \textcircled{\underline{a}}^\delta \\ \textcircled{(\underline{a}, \underline{b})} \\ \textcircled{\underline{b}}^\delta \end{array} \right] \quad (15)$$

Grounded nodes are root nodes (i.e., nodes with no incoming arrows, only outgoing ones) which have a deterministic probability amplitude (i.e., an amplitude that equals 1 for just one of the possible states of the node and zero for all the other states). Grounded nodes will be indicated by writing a zero near them. Here is an example of a QB net with a grounded node:

$$\left[\begin{array}{c} \textcircled{\underline{a}} \\ \textcircled{(\underline{a}, \underline{b})} \\ \textcircled{\underline{b}}^0 \end{array} \right], \quad (16)$$

where³

$$A(b) = \delta_b^0, \quad (17)$$

for all $b \in S_{\underline{b}}$.

6 Isometries

Consider the following QB net

$$\left[\textcircled{\underline{b}} \leftarrow \textcircled{\underline{a}} \right], \quad (18)$$

where $A(b|a)$ satisfies

$$\sum_{b \in S_{\underline{b}}} [A(b|a)] \left[\begin{array}{c} \text{h.c.} \\ a \rightarrow a' \end{array} \right] = \delta_a^{a'}, \quad (19)$$

³ We are assuming that $0 \in S_{\underline{b}}$. It doesn't matter if the amplitude $A(b)$ of node \underline{b} equals δ_b^0 , or $\delta_b^{b_0}$ where $b_0 \in S_{\underline{b}}$. Either way, it's still a grounded node. An alternative to writing a zero next to a node to indicate that it's grounded might be writing instead the letters "grd" or the electrical symbol for a ground.

for all $a, a' \in S_{\underline{a}}$. The node \underline{b} in the above QB net is called an **isometry node**, or just an isometry for simplicity.

Eq.(19) is saying that the column vectors of the transition matrix $A(b|a)$ are orthonormal. This is only possible if $N_{\underline{b}} \geq N_{\underline{a}}$. If $N_{\underline{b}} = N_{\underline{a}}$ (i.e., transition matrix is square), then the transition matrix is unitary. If $N_{\underline{b}} > N_{\underline{a}}$ (i.e., transition matrix is rectangular with more rows than columns), then we can use the well-known, so called Gram-Schmidt procedure to add more columns to the transition matrix (“extend it”) to produce a unitary matrix.

Since $N_{\underline{b}} \geq N_{\underline{a}}$ and the sets $S_{\underline{b}}, S_{\underline{a}}$ are finite, we may assume without loss of generality that $S_{\underline{b}} \supset S_{\underline{a}}$. For every $b, A \in S_{\underline{b}}$, let⁴

$$A(b|A) = \begin{cases} A(b|a) & \text{if } A = a \in S_{\underline{a}} \\ \text{given by Gram-Schmidt procedure} & \text{if } A \in S_{\underline{b}} - S_{\underline{a}} \end{cases} \quad (20)$$

Then

$$\sum_{b \in S_{\underline{b}}} [A(b|A)] \begin{bmatrix} \text{h.c.} \\ A \rightarrow A' \end{bmatrix} = \delta_{A'}^A \quad (21)$$

for all $A, A' \in S_{\underline{b}}$. Thus,

$$A(b|A) = \langle b|U_{\underline{b}}|A \rangle, \quad (22)$$

where $U_{\underline{b}}$ is unitary.

Here is a pictorial representation, in terms of QB nets, of the procedure just outlined for extending an isometry to a unitary matrix:

$$\left[\begin{array}{c} \textcircled{\underline{b}} \longleftarrow \textcircled{\underline{a}} \end{array} \right] \rightarrow \left[\begin{array}{c} \textcircled{\underline{b}} \longleftarrow \textcircled{\underline{A}} \end{array} \right], \quad (23)$$

where $S_{\underline{b}} = S_{\underline{A}} \supset S_{\underline{a}}$.

Consider the following QB net

$$\left[\begin{array}{c} \textcircled{(\underline{a}, \underline{b})} \longleftarrow \textcircled{\underline{a}} \end{array} \right], \quad (24)$$

where $A(a, b|a')$ satisfies

$$\sum_{a \in S_{\underline{a}}} \sum_{b \in S_{\underline{b}}} [A(a, b|a')] \begin{bmatrix} \text{h.c.} \\ a' \rightarrow a'' \end{bmatrix} = \delta_{a''}^{a'}, \quad (25)$$

for all $a', a'' \in S_{\underline{a}}$. The node $(\underline{a}, \underline{b})$ in the above QB net is a special case of the general isometry node presented previously. Just as in the case of a general isometry, the

⁴We are using the symbol A both for an element A of $S_{\underline{b}}$, and for amplitudes $A(\cdot)$. It's easy to tell which usage is intended in each instance, so this should cause no confusion.

transition matrix $A(a, b|a')$ can be extended to a unitary matrix. Assume $0 \in S_{\underline{b}}$. If we define

$$A(a, b|a', b' = 0) = A(a, b|a') , \quad (26)$$

then we can find a unitary matrix $U_{\underline{a}, \underline{b}}$ such that, for all $a, a' \in S_{\underline{a}}$ and $b, b' \in S_{\underline{b}}$,

$$A(a, b|a', b') = \begin{array}{cc} \langle a|_{\underline{a}} & |a'\rangle_{\underline{a}} \\ U_{\underline{a}, \underline{b}} & \\ \langle b|_{\underline{b}} & |b'\rangle_{\underline{b}} \end{array} . \quad (27)$$

Here is a pictorial representation, in terms of QB nets, of the procedure just outlined for extending an isometry to a unitary matrix:

$$\left[\begin{array}{c} \textcircled{(\underline{a}, \underline{b})} \leftarrow \textcircled{\underline{a}} \end{array} \right] = \left[\begin{array}{c} \textcircled{\underline{a}} \\ \swarrow \searrow \\ \textcircled{(\underline{a}, \underline{b})} \\ \swarrow \searrow \\ \textcircled{\underline{b}} \end{array} \right] \rightarrow \left[\begin{array}{c} \textcircled{\underline{a}} \\ \swarrow \searrow \\ \textcircled{(\underline{a}, \underline{b})} \\ \swarrow \searrow \\ \textcircled{\underline{b}} \end{array} \right] \quad (28)$$

7 Freeing a Bound Index

We've defined the QB net corresponding to the meta ket state as having for each node \underline{b} : (1) an index b that is summed over (bound), and (2) a ket $|b\rangle_{\underline{b}}$. One can free an index b of a node \underline{b} of a meta ket state by multiplying that node by $\langle b|$ or $|b\rangle\langle b|$. One can use QB nets to represent these two operations. For example, if the meta state is,

$$\left[\textcircled{\underline{b}} \leftarrow \textcircled{\underline{a}} \right] = \sum_{a,b} A(b|a) A(a) |b\rangle_{\underline{b}} |a\rangle_{\underline{a}} , \quad (29)$$

then

$$\left[\textcircled{\langle b|} \textcircled{\underline{b}} \leftarrow \textcircled{\underline{a}} \right] = \sum_a A(b|a) A(a) |a\rangle_{\underline{a}} , \quad (30)$$

and

$$\left[\textcircled{|b\rangle\langle b|} \textcircled{\underline{b}} \leftarrow \textcircled{\underline{a}} \right] = |b\rangle_{\underline{b}} \sum_a A(b|a) A(a) |a\rangle_{\underline{a}} . \quad (31)$$

8 Classical Communication

Quantum information theorists call “classical communication” the act of measuring an observable at one event and then using the result of that measurement to start a new event. Classical communication can be represented using QB nets. For example, if $S_{\underline{c}} = S_{\underline{b}}$, then

$$\left[\begin{array}{c} \textcircled{\underline{d}} \xleftarrow{\langle b|} \textcircled{\underline{c}} \\ \textcircled{\underline{b}} \xleftarrow{\langle b|} \textcircled{\underline{a}} \end{array} \right] = \sum_{d,a} A_{d|\underline{c}}(d|b) A_{\underline{c}}(b) A_{\underline{b}|\underline{a}}(b|a) A_{\underline{a}}(a) |d\rangle_{\underline{d}} |a\rangle_{\underline{a}} . \quad (32)$$

9 (Coherent or Incoherent)-(Scalar or Vector) Sums

For any $\rho_{\underline{b},\underline{a}} \in dm(\mathcal{H}_{\underline{b},\underline{a}})$, define

$$\text{tr}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \rho_{\underline{a}} = \sum_b \left[\langle b|_{\underline{b}} \right] \rho_{\underline{b},\underline{a}} \left[\text{h.c.} \right] , \quad (33)$$

$$\text{cl}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \rho_{\underline{b},\underline{a}} = \sum_b \left[|b\rangle_{\underline{b}} \langle b|_{\underline{b}} \right] \rho_{\underline{b},\underline{a}} \left[\text{h.c.} \right] , \quad (34)$$

$$\text{sl}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \rho_{\underline{b},\underline{a}} = \left[\sum_b \langle b|_{\underline{b}} \right] \rho_{\underline{b},\underline{a}} \left[\text{h.c.} \right] . \quad (35)$$

Note that

$$\text{tr}_{\underline{b}}(1) = N_{\underline{b}}, \quad \text{cl}_{\underline{b}}(1) = 1, \quad \text{sl}_{\underline{b}}(1) = N_{\underline{b}} . \quad (36)$$

Note also that the product of any two operators in the set $F = \{1, \text{tr}_{\underline{b}}, \text{cl}_{\underline{b}}, \text{sl}_{\underline{b}}\}$ can be expressed in terms of a single one of them. For example,

$$\text{tr}_{\underline{b}} \text{cl}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \text{tr}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) , \quad \text{tr}_{\underline{b}} \text{tr}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = N_{\underline{b}} \text{tr}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) , \quad (37)$$

etc.. Hence, a product of any number of the operators in F can be expressed in terms of a single one of them.⁵

For each node \underline{b} of a meta density matrix, there is an index b that is summed over and a ket $|b\rangle_{\underline{b}}$. Furthermore, the \sum_b is inside the sandbox, so we say that it's a **coherent sum**. Because the term being summed (i.e., the summand) includes the ket $|b\rangle_{\underline{b}}$, we say it's a **vector sum**.

If the \sum_b were outside the sandbox (and index b appeared in both the sandbox and its dual), we would call it an **incoherent sum**. If the summand did not include $|b\rangle_{\underline{b}}$, we would call it a **scalar sum**. The operators $\text{tr}_{\underline{b}}(\cdot)$, $\text{cl}_{\underline{b}}(\cdot)$, $\text{sl}_{\underline{b}}(\cdot)$ act on the meta

⁵More formally, if we define $c = \text{cl}_{\underline{b}}$, $\sigma = \text{sl}_{\underline{b}}/N_{\underline{b}}$, $\tau = \text{tr}_{\underline{b}}/N_{\underline{b}}$, and $F = \{1, c, \sigma, \tau\}$, then it is easy to check that for all $f \in F$, $f\tau = \tau$, $f\sigma = \sigma$, and $fc = \begin{cases} c & \text{if } f \in \{1, c\} \\ \tau & \text{otherwise} \end{cases}$. Although F is closed under composition, it is not a group. This is not surprising since c, ρ, σ are irreversible transformations.

density matrix to change the coherent-vector sum over a node \underline{b} to an incoherent-scalar, or an incoherent-vector, or a coherent-scalar sum. Let's illustrate this with an example.

- Consider the following meta density matrix as an example. Note that in this meta density matrix, for the random variable \underline{b} , there is a **coherent-vector sum** over the index b .

$$\rho_{\underline{b},\underline{a}} = \left[\sum_{a,b} A(b|a) A(a) |b\rangle_{\underline{b}} |a\rangle_{\underline{a}} \right] [\text{h.c.}] \quad (38a)$$

$$= \left[\textcircled{\underline{b}} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (38b)$$

- “Tracing” (i.e., taking a partial trace of) the random variable \underline{b} means doing an **incoherent-scalar sum** over the index b .

$$\text{tr}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \rho_{\underline{a}} = \sum_b \left[\sum_a A(b|a) A(a) |a\rangle_{\underline{a}} \right] [\text{h.c.}] \quad (39a)$$

$$= \left[\textcircled{\underline{b}}^{tr} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (39b)$$

$$= \sum_b \left[\langle b| \textcircled{\underline{b}} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (39c)$$

- “Classicizing”, or “Making classical” the random variable \underline{b} means doing an **incoherent-vector sum** over the index b . (This operation is also sometimes described as “dephasing” because we are throwing away some off-diagonal terms).

$$\text{cl}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \rho_{\underline{b},\underline{a}}^{cl} = \sum_b \left[\sum_a A(b|a) A(a) |b\rangle_{\underline{b}} |a\rangle_{\underline{a}} \right] [\text{h.c.}] \quad (40a)$$

$$= \left[\textcircled{\underline{b}}^{cl} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (40b)$$

$$= \sum_b \left[|b\rangle\langle b| \textcircled{\underline{b}} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (40c)$$

- “Slashing” the random variable \underline{b} means doing a **coherent-scalar sum** over the index b .

$$\text{sl}_{\underline{b}}(\rho_{\underline{b},\underline{a}}) = \rho_{\underline{b},\underline{a}}^{sl} = \left[\sum_{a,b} A(b|a) A(a) |a\rangle_{\underline{a}} \right] [\text{h.c.}] \quad (41a)$$

$$= \left[\textcircled{\cancel{\underline{b}}} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (41b)$$

$$= \left[\sum_b \langle b| \textcircled{\underline{b}} \longleftarrow \textcircled{\underline{a}} \right] [\text{h.c.}] \quad (41c)$$

Note that the operators in F are all irreversible transformations (except for the 1). The meta state is truly “at the top of the food chain”: Once the operators in F take the meta state to something else, no operator or combination of operators in F can bring back the same meta state.

10 Ensembles, Purification

An **ensemble** is a set $\{\sqrt{w_j}|\psi_j\rangle_{\underline{x}}\}_{\forall j}$ where the weights w_j are non-negative numbers that sum to 1, and for all j , the states $|\psi_j\rangle_{\underline{x}} \in \mathcal{H}_{\underline{x}}$ are normalized but they are not necessarily mutually orthogonal. The density matrix for this ensemble is

$$\rho_{\underline{x}} = \sum_j w_j [|\psi_j\rangle_{\underline{x}}] [\text{h.c.}] . \quad (42)$$

Define two ensembles as being equivalent if they have the same density matrix. This defines an equivalence relation. Elements of the same equivalence class are physically indistinguishable.

The density matrix Eq.(42) can be **purified**, meaning that it can be expressed as a partial trace of a pure state. One way of doing this is as follows. Clearly, $\rho_{\underline{x}}$ also equals

$$\rho_{\underline{x}} = \text{tr}_{\underline{j}} \left[\frac{\sum_x |x\rangle \langle x| \psi_j\rangle_{\underline{x}}}{\sum_j \sqrt{w_j} |j\rangle_{\underline{j}}} \right] [\text{h.c.}] . \quad (43)$$

Thus

$$\rho_{\underline{x}} = \text{tr}_{\underline{j}} [|\psi\rangle_{\underline{x}, \underline{j}}] [\text{h.c.}] , \quad (44)$$

where

$$|\psi\rangle_{\underline{x}, \underline{j}} = \left[\sum_{x,j} A(x,j) \frac{|x\rangle_{\underline{x}}}{|j\rangle_{\underline{j}}} \right] = \left[\textcircled{\underline{x}} \longleftarrow \textcircled{\underline{j}} \right] , \quad (45)$$

where

$$A(x,j) = A(x|j)A(j) , \quad (46)$$

and

$$A(x|j) = \langle x|\psi_j\rangle, \quad A(j) = \sqrt{w_j} . \quad (47)$$

11 Measurement Superoperators

A **superoperator** is a linear operator that maps $dm(\mathcal{H}_{\underline{a}})$ into $dm(\mathcal{H}_{\underline{b}})$.

A **measurement** is defined as a set $\{K_\mu | \mu \in S_\mu\}$ of operators K_μ called **Krauss operators** that map states in \mathcal{H}_a to states in \mathcal{H}_b . We assume $N_a \leq N_b$. ($N_a = |S_a| = \dim(\mathcal{H}_a)$ and the same for b). The Krauss operators must also satisfy:

$$\sum_{\mu} K_\mu^\dagger K_\mu = 1 . \quad (48)$$

Each Krauss operator K_μ can be used to define a superoperator $\$_\mu(\cdot)$ as follows. Let $\rho_a \in dm(\mathcal{H}_a)$, and, for each μ , let $\sigma_{b|\mu} \in dm(\mathcal{H}_b)$. Then define the **measurement superoperator** $\$_\mu(\cdot)$ by

$$\$_\mu(\rho_a) = \rho_{b|\mu} = \frac{K_\mu \rho_a K_\mu^\dagger}{P(\mu)} , \quad (49)$$

where

$$P(\mu) = \text{tr}_b(K_\mu \rho_a K_\mu^\dagger) = \text{tr}_a(K_\mu^\dagger K_\mu \rho_a) . \quad (50)$$

Note that the $P(\mu)$ are non-negative and

$$\sum_{\mu} P(\mu) = 1 . \quad (51)$$

Note also that

$$\text{tr}_b(\rho_{b|\mu}) = 1 \quad (52)$$

for all μ .

A **von Neumann measurement** (for instance, $K_\mu = |\mu\rangle\langle\mu|$) is a measurement $\{K_\mu\}_{\forall\mu}$ that satisfies:

$$K_\mu^\dagger = K_\mu, \quad K_\mu K_{\mu'} = \delta_{\mu}^{\mu'}, \quad \sum_{\mu} K_\mu = 1 . \quad (53)$$

Here are some other examples of measurements (you can check that $\sum_a K_a^\dagger K_a = 1$ for each example)

- Tracing: $K_a = \langle a|_a$
- Making a node classical (i.e., dephasing it): $K_a = |a\rangle_a \langle a|_a$
- Classical (incoherent) communication: $K_a = |a\rangle_b \langle a|_a$, where $S_a = S_b$.
- Coherent communication (only one Krauss operator): $K = \sum_a |a\rangle_b \langle a|_a$, where $S_a = S_b$.

A measurement $\{K_\mu\}_{\forall\mu}$ can be extended to a unitary operator as follows. For every $b \in S_{\underline{b}}$, $\mu \in S_{\underline{\mu}}$, $a \in S_{\underline{a}}$, define

$$A(b, \mu|a) = \langle b|K_\mu|a \rangle . \quad (54)$$

Since for all $a, a' \in S_{\underline{a}}$,

$$\sum_{b,\mu} [A(b, \mu|a)] \left[\begin{array}{c} \text{h.c.} \\ a \rightarrow a' \end{array} \right] = \sum_{b,\mu} \langle b|K_\mu|a \rangle \langle a'|K_\mu^\dagger|b \rangle = \delta_{a'}^a , \quad (55)$$

$A(b, \mu|a)$ defines an isometry. Assume $0 \in S_{\underline{\mu}}$. Let

$$A(b, \mu|a, \mu' = 0) = A(b, \mu|a) . \quad (56)$$

Since $N_{\underline{a}} \leq N_{\underline{b}}$, we can use Gram Schmidt to find a unitary operator $U_{\underline{b},\underline{\mu}}$ such that

$$A(b, \mu|A, \mu') = \begin{array}{cc} \langle b|_{\underline{b}} & |A\rangle_{\underline{A}} \\ & U_{\underline{b},\underline{\mu}} \\ \langle \mu|_{\underline{\mu}} & |\mu'\rangle_{\underline{\mu}} \end{array} \quad (57)$$

for all $b, A \in S_{\underline{b}} = S_{\underline{A}}$ and $\mu, \mu' \in S_{\underline{\mu}}$.

12 RINNO (POVM)

A POVM, which I prefer to call a RINNO, is a Resolution of the Identity by Non Negative Operators. Thus, a RINNO $\{R_\mu\}_{\forall\mu}$ satisfies

$$\sum_{\mu} R_\mu = 1, \quad R_\mu \geq 0 . \quad (58)$$

(Each R_μ is a square matrix. A square matrix M is said to be non-negative, or said to satisfy $M \geq 0$, if $v^\dagger M v \geq 0$ for all complex column vectors v .)

Suppose $\rho_{\underline{a}} \in dm(\mathcal{H}_{\underline{a}})$, and for each μ , R_μ maps $\mathcal{H}_{\underline{a}}$ into itself. For each μ , define

$$P(\mu) = \text{tr}_{\underline{a}}(R_\mu \rho_{\underline{a}}) . \quad (59)$$

By Eqs.(58), the $P(\mu)$ are non-negative and satisfy $\sum_{\mu} P(\mu) = 1$.

A RINNO $\{R_\mu\}_{\forall\mu}$ can be constructed from a measurement $\{K_\mu\}_{\forall\mu}$ by setting

$$R_\mu = K_\mu^\dagger K_\mu \quad (60)$$

for each μ . The definition of a measurement $\{K_\mu\}_{\forall\mu}$ and Eqs.(60) imply Eqs.(58).

13 Channel Superoperators

Suppose $\{K_\mu | \mu \in S_\mu\}$ is a measurement with Krauss operators $K_\mu : \mathcal{H}_a \rightarrow \mathcal{H}_b$. Let $\rho_a \in dm(\mathcal{H}_a)$ and $\sigma_b \in dm(\mathcal{H}_b)$. Then define the **channel superoperator** $\$(\cdot)$ by⁶

$$\$(\rho_a) = \sigma_b = \sum_{\mu} K_\mu \rho_a K_\mu^\dagger. \quad (61)$$

Note that a channel superoperator is a weighted sum of measurement superoperators (i.e., $\$ = \sum_{\mu} P(\mu) \$_\mu$).

ρ_a can always be expressed as

$$\rho_a = \sum_j w_j [|\psi_j\rangle_a] [\text{h.c.}], \quad (62)$$

where the weights $\{w_j\}_{\forall j}$ are non-negative numbers that sum to one, and the states $\{|\psi_j\rangle_a\}_{\forall j}$ are all normalized but not necessarily mutually orthogonal. Note that for all $b, b' \in S_b$,

$$\langle b | \sigma_b | b' \rangle = \sum_{\mu, j} [\langle b | K_\mu | \psi_j \rangle \sqrt{w_j}] \left[\begin{array}{c} \text{h.c.} \\ b \rightarrow b' \end{array} \right] \quad (63a)$$

$$= \sum_{\mu, j} \left[\begin{array}{cc} \langle b |_b & |\psi_j\rangle_a \\ \langle \mu |_\mu & |0\rangle_\mu \end{array} U_{b, \mu} \sqrt{w_j} \right] \left[\begin{array}{c} \text{h.c.} \\ b \rightarrow b' \end{array} \right] \quad (63b)$$

where, as discussed in Section 11, $U_{b, \mu}$ is a unitary matrix that extends the measurement $\{K_\mu | \mu \in S_\mu\}$.

Eq.(62) for ρ_a and Eq.(63b) for σ_b can be represented as follows in terms of QB nets:

$$\rho_a = \text{tr}_j \left[\begin{array}{c} \textcircled{a} \leftarrow \textcircled{j} \end{array} \right] [\text{h.c.}], \quad (64)$$

$$\sigma_b = \text{tr}_{\mu, j} \left[\begin{array}{ccc} \textcircled{b}^\delta & & \textcircled{A} \leftarrow \textcircled{j} \\ & \nwarrow \quad \nearrow & \\ & \textcircled{(b, \mu)} & \\ & \swarrow \quad \searrow & \\ \textcircled{\mu}^\delta & & \textcircled{\mu}^0 \end{array} \right] [\text{h.c.}], \quad (65)$$

⁶ Krauss showed that for any superoperator $\$(\cdot)$, $\$(\cdot)$ is a channel superoperator iff $\$(\cdot)$ is “completely positive”.

where

$$A(\mu) = \delta_\mu^0, \quad (66)$$

$$A(j) = \sqrt{w_j}, \quad (67)$$

$$A(A|j) = \begin{cases} \langle A|\psi_j\rangle & \text{if } A \in S_{\underline{a}} \\ 0 & \text{if } A \in S_{\underline{b}} - S_{\underline{a}} \end{cases}, \quad (68)$$

$$A(b, \mu|A, \mu') = \frac{\langle b|_{\underline{b}} \quad |A\rangle_{\underline{A}}}{\langle \mu|_{\underline{\mu}} \quad |\mu'\rangle_{\underline{\mu}}} U_{\underline{b}, \underline{\mu}}, \quad (69)$$

$$A(b|b', \mu') = \delta_b^{b'}, \quad (70)$$

$$A(\mu|b', \mu') = \delta_\mu^{\mu'}. \quad (71)$$

14 Complementary Channel

The channel superoperator $\$(\cdot)$ given by Eq.(61) can be used to define a **complementary channel** superoperator $\$'(\cdot)$. If $\$(\cdot)$ is generated using a measurement $\{K_\mu\}_{\forall \mu}$, then we can find a unitary operator $U_{\underline{b}, \underline{\mu}}$ such that

$$K_\mu^{b \leftarrow A} = \frac{\langle \mu|_{\underline{\mu}} \quad |0\rangle_{\underline{\mu}}}{\langle \mu|_{\underline{\mu}} \quad |0\rangle_{\underline{\mu}}} U_{\underline{b}, \underline{\mu}}. \quad (72)$$

Now define a measurement $\{L_b\}_{\forall b}$ using the same unitary operator $U_{\underline{b}, \underline{\mu}}$:

$$L_b^{\mu \leftarrow \underline{\mu}} = \frac{\langle b|_{\underline{b}} \quad |0\rangle_{\underline{A}}}{\langle b|_{\underline{b}} \quad |0\rangle_{\underline{A}}} U_{\underline{b}, \underline{\mu}}. \quad (73)$$

Then

$$\sigma_{\underline{b}} = \$(\rho_{\underline{a}}) = \sum_{\mu} K_\mu \rho_{\underline{a}} K_\mu^\dagger \quad (74)$$

and

$$\sigma_{\underline{\mu}} = \$'(\rho_{\underline{\mu}}) = \sum_b L_b \rho_{\underline{\mu}} L_b^\dagger. \quad (75)$$

We've already shown how density matrices $\rho_{\underline{a}}$ and $\sigma_{\underline{b}} = \$(\rho_{\underline{a}})$ can be represented by QB nets (see Eqs.(64) and (65)). Likewise, density matrices ρ_{μ} and $\sigma_{\mu} = \$(\rho_{\mu})$ can be represented by QB nets as follows:

$$\rho_{\underline{\mu}} = \text{tr}_{\underline{j}} \left[\left(\textcircled{\underline{\mu}} \leftarrow \textcircled{\underline{j}} \right) \right] [\text{ h.c. }] \quad (76)$$

$$\sigma_{\underline{\mu}} = \text{tr}_{\underline{b}, \underline{j}} \left[\begin{array}{c} \text{Diagram: A central node } (\underline{b}, \underline{\mu}) \text{ with four outgoing arrows to nodes } \underline{\not{b}}, \underline{\not{A}}, \underline{\mu}, \text{ and } \underline{\mu}. \text{ The node } \underline{\not{b}} \text{ has a label } \delta. \text{ The node } \underline{\not{A}} \text{ has a label } 0. \text{ The node } \underline{\mu} \text{ at the bottom left has a label } \delta. \text{ The node } \underline{\mu} \text{ at the bottom right has an incoming arrow from a node } \underline{j}. \end{array} \right] [\text{h.c.}] \quad (77)$$

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